Math 3305 Chapter 1 Section 1.1

Opening notes: Section 1.1 has 3 videos and all three scripts are in this one document. Popper 1.1 goes throughout the 3 videos and is 12 questions long. Be sure to finish and submit your answers by the deadline. There are no homework problems in 1.1; those pick up in 1.2. When you get to them, they will be turned in with all the homework problems from 1.3, 1.4 and 1.5 in one document under “Homework Chapter 1” under the assignment tab in CourseWare. There are two essays (One, page 16 and Two, page 34) will be turned in individually also under the assignments tab; one per assignment; the essays are discussed in the videos and are written out in this document. Be sure to stick to one page. PDF them and turn them in by the deadline. Deadlines are in the Course Calendar. NOW!

Video 1 for 1.1. The Ancient Stuff and some history.

We begin with the STRUCTURE of geometry. A geometry is an axiomatic system. Note “A” not “The”.

Some other geometries in additional to traditional Euclidean Geometry:

Big Ones: Spherical, Hyperbolic, Finsler, …

Little Ones: Triangle, Klein, Fano’s,…

Areas of study other than Geometry have an axiomatic structure. Probability and Logic for example, and Real Analysis. We will restrict ourselves to Geometry and it’s axiomatic structure.

Modern axiomatic system parts:

Undefined terms

Axioms

Definitions

Theorems – lemmas and corollaries

The first axiomatic system was devised by Euclid and he recorded it in his book Elements centuries before Christ was born. It was WAY shorter than what we use now. Let’s look at it briefly and I’ll point out some problems that have been noticed and fixed over the centuries since his time.

Euclid’s system:

Common Notions

Definitions

Postulate

Theorems

Definitions

He began with Common Notions. These are now reduced to vocabulary and vocabulary. We no longer list the information BEHIND the system. Most of those have their own axiomatic systems now.

CN1 Things that are equal to the same thing are also equal to each other.

The transitive property of equality – Equivalence Relations in 2.1

CN2 If equals are added to equals, the sums are equal.

CN3 If equals are subtracted from equals, the remainders are equal.

CN4 Things that coincide with each other are equal to each other.

Congruent

CN5 The whole is greater than the part.

He then went on to 26 definitions. Let’s look at a couple:

A point is that which has no part.

A line is a breadthless length.

The ends of a line are points.

A straight line is a line which lies evenly with the points on itself.

A surface is that which has length and breadth only.

The edges of a surface are lines.

We don’t do this any more. The undefined terms in the axiomatic system I’ll teach you in the next video are: point, line, half-plane, and plane. We will have MANY definitions but every geometry starts with undefined terms now. And fewer is considered better now.

He then went on to his Postulates (now called axioms).

P1 You may draw a straight line from any point to any point.

P2 You may draw a straight line continuously from one line to another.

P3 You may draw a circle with any center and radius.

P4 Know that each right angle is equal to each other right angle.

P5 If a straight line crossing two other straight line has angles on one side less that right angles, then those two lines crossed meet on that side somewhere.

In the next video: 22 axioms. In Birkhoff: 5 axioms. In middle school books sometimes 40 or 50 axioms!

**Popper 1.1, Question One**

Euclid began what is now known as an Axiomatic System centuries ago.

A. True

B. False

Then Theorems – 13 books of theorems! Let’s just look at the first two. These are constructions. Done occasionally now. We have rulers, though, and protractors. Euclid didn’t have those. And he only had natural numbers!

Theorem 1.1 You may construct an equilateral triangle on a given straight line.

Theorem 1.2 You may place a straight line equal to a given straight line on the end of another given line.

This finishes up the ancient part of our look at Geometry. Many people like to hark back to Euclid’s axioms when talking geometry but I think it’s MUCH better to go with modern axioms and be totally complete with our ideas and statements.

Video 2 for Chapter 1 Section 1.

Mostly new, more modern information. Popper 1.1 continues.

From 1958 – 1977 the Schoolhouse Mathematics Study Group met regularly to work on producing a study guide for a national curriculum at the high school level for Euclidean Geometry. They published the following in 1961.

## The SMSG Axioms for Euclidean Geometry

Undefined terms

Axioms

Theorems and Definitions came along later.

SMSG Undefined Terms for Euclidean Geometry:

point, line, half plane, plane, and space

We take these as our beginning point. We can visualize, sketch, or model, just not define.

Most people visualize a point as a tiny, tiny dot. Lines are thought of as long, seamless concatenations of points and planes are composed of finely interwoven lines: smooth, endless and flat.

Think of undefined terms as the basic sounds in a language – the sounds that make up our language for the most part have no meaning in themselves but are combined to make words.

The grammar of our language and a good dictionary are what make the meaning of the sounds. This part of language corresponds to the axioms and definitions that you will find as we move along in the sections and chapters.

From these the facts, flights of fancy, and content-laden sentences are built – these are the theorems and definition in an axiomatic system.

The conventions of the Cartesian plane (17th century, Rene Descartes), the most common visualization of Euclidean Geometry, are well suited to assisting in “seeing” and working with Euclidean geometry. However there are some differences between a geometric approach to points on a line and an algebraic one, as we will see in the explanation of Axiom 3.

SMSG Axioms for Euclidean Geometry:

The axioms are grouped in 5 sections. The first eight axioms deal with points, lines, planes, and distance. Then comes convexity and separation issues – these two axioms deal with facts about the relationships among our undefined objects on a set theoretic basis. Axioms 11 through 14 introduce angles: measuring and constructing them as well as some fundamental facts about linear pairs. With Axiom 15, we begin to look at congruent triangles – note that this is so fundamental a notion that it requires its own axiom. Axiom 16 introduces parallel lines. We then look at area for polygons and congruent triangles (axioms 17 – 20) and we finish up with two axioms about solid figures.

First some words about axioms. Axioms in a system are always true; and they never contradict one another. Axioms discuss the relationships among the undefined terms and they expand what can be done legally in the system. They are the “rules to the game”. They vary greatly from system to system as you will see in video three.

**Popper 1.1, Question Two**

Which of the following are part of a modern Axiomatic System?

I. Undefined Terms

II. Axioms

III. Common notions

IV. Theorems

V. Definitions

VI. Guesses

A. All of these

B. III and VI only

C. I, II, and III

D. I, II, IV, and V

E. None of these

SMSG Axioms for Euclidean Geometry:

Group 1

A1. Given any two distinct points there is exactly one line that contains them.

Exactly the same as with Euclid.

A2. The Distance Postulate:

To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

A3. The Ruler Postulate:

The points of a line can be placed in a correspondence with the real numbers such that

A. To every point of the line there corresponds exactly one real number.

B. To every real number there corresponds exactly one point of the line,

and

C. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

A4. The Ruler Placement Postulate:

Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

**Popper 1.1, Question Three**

SMSG Axiom 4 refers only to the standard horizontal x-axis number line.

A. True

B. False

A5. A. Every plane contains at least three non-collinear points.

B. Space contains at least four non-coplanar points.

**Popper 1.1, Question Four**

“Plane”, “point”, and “space” are undefined terms in this geometry. SMSG Axiom 5 is describing facts and relationships about and among these terms.

A. True

B. False

A6. If two points line in a plane, then the line containing these points lies in the same plane.

A7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

A8. If two planes intersect, then that intersection is a line.

Group 2

A9. The Plane Separation Postulate:

Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

A. each of the sets is convex, and

B. if P is in one set and Q is in the other, then segment PQ intersects the

line.

Convexity:

Viz of B

A10. The Space Separation Postulate:

#### The points of space that do not lie in a given plane form two sets such that

A. each of the sets is convex, and

B. if P is in one set and Q is in the other, then the segment PQ intersects

the plane.

Group 3

A11. The Angle Measurement Postulate:

To every angle there corresponds a real number between 0° and 180°.

A12. The Angle Construction Postulate:

Let AB be a ray on the edge of the half-plane H. For every r between 0 and 180 there is exactly one ray AP with P in H such that m ∠ PAB = r.

A13. The Angle Addition Postulate:

If D is a point in the interior of ∠ BAC, then

m ∠ BAC = m ∠ BAD + m ∠ DAC.

A14. The Supplement Postulate:

If two angles form a linear pair, then they are supplementary

Group 4

A15. The SAS Postulate:

Given an one-to-one correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

Since Euclid’s days we have discovered that there are several major categories of geometries. There are geometries that have a finite number of points and no parallel lines, or more than one line parallel to the given line.

We will look at a couple of these in the next video and as we move through the semester.

There are the Big Three: Euclidean, Spherical and Hyperbolic. Each has it’s own axiom system and each is distinctly different about parallel lines.

We will pick a line and a point not on that line and see that in

Euclidean: exactly one that shares no points

Spherical: no parallel lines

Hyperbolic: more than one line parallel to the given line through the external point.

Group 5

A17. To every polygonal region there corresponds a unique positive number called its area.

A18. If two triangles are congruent, then the triangular regions have the same area.

A19. Suppose that the region R is the union of two regions R1 and R2. If R1 and R2 intersect at most in a finite number of segments and points, then the area of R is the sum of the areas of R1 and R2.

A20. The area of a rectangle is the product of the length of its base and the length of its altitude.

21. The volume of a rectangular parallelpiped is equal to the product of the length of its altitude and the area of its base.

A22. Cavalieri’s Principal:

Given two solids and a plane. If for every plane that intersects the solids and is parallel to the given plane, the two intersections determine regions that have the same area, then the two solids have the same volume.

**Essay 1.1 Number One**

What are axioms and why do we care about them?

One page, front side only, 12 point type. PDF and turn in under this title in Assignments in CourseWare.

**Popper 1.1, Question Five**

Sometimes there are formulas in the axioms.

A. True

B. False

This wraps up the modern axiomatic system portion of section 1.1. See you in Video 3! Next up.

Video 3 Chapter 1 Section 1 Other Geometries!

Other Geometries

Timeline – 19th century

Context!

NonEuclidean Geometries

3 Point

Flexible

7 Point, Fano’s

Klein Disc

Spherical

Hyperbolic

The Three-point Geometry

Undefined Terms: point, line, on

Axioms: A1 There are exactly 3 distinct points

A2 Two distinct points are on exactly one line.

A3 Not all the points are on the same line.

A4. Each pair of distinct lines are on at least one point.

Definitions: Lines that share a point are intersecting.

Lines that do not share a point are parallel.

Model:

 Label the points A, B, and C

Check the axioms and definition.

What is exactly 1/3 of the way between B and C?

Impossible? Draw me a unicorn.

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

Theorem 1: Each pair of distinct lines is on exactly one point.

Proof of Theorem 1

Suppose there’s a pair of lines on more than one point. This cannot be because then the two lines have at least two distinct points on each of them and Axiom 1 states that “two distinct points are on exactly one line”.

Thus our supposition cannot be and the theorem is true. QED

Theorem 2: There are exactly 3 distinct lines in this geometry.

**Popper 1.1, Question 6**

Some geometries have lines that are not made up of an infinite number of points.

A. True

B. False

One of the tinier ones. A Flexible Geometry is smaller…just 2 axioms! More more complicated models. Let’s think about the Three Point just a little longer…

Could this geometry be called non-Euclidean? Let’s come up with some reasons why?

Why is this a geometry?

Let’s do these two together

**Popper 1.1, Question Seven**

Why is this a geometry? It has the right structure and the right kind of undefined terms.

A.\* True

**Popper 1.1, Question Eight**

There are axiomatic structures for other math systems like probability and logic. They have the same structure and different undefined terms.

A.\* True

**The Seven-point geometry**

Also known as Fano’s geometry. (Gino Fano, published 1892, 1871 – 1952, Italian)

Undefined terms: point, line, on

A1: There exists at least one line.

A2: There exist exactly three points on each line.

A3: Not all the points are on the same line.

A4: There exists exactly one line on any two distinct points.

A5: There exists at least one point on any two distinct lines.



Model: How many points and lines?

{BDF} is a line! Nobody said “straight” in the axioms!

Where does {BDF} intersect {CBA}? Not to mention:

How many points on each line?

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

7 points and 7 lines…what’s the situation with respect to parallel lines?



Alternate axioms for Fano’s Geometry:

Undefined terms: point, line, on

Axioms:

A1 For every pair of distinct points P and Q, there exists exactly one line *l* such that both P and Q lie on that line.

Note that the axiom uses all 3 undefined terms and is defining a relationship among them.

A2 For every line *l* there exist at least 2 distinct points P and Q such that both P and Q lie on the line *l*.

A3 There exist three points that do not all lie on any one line.

Note the “at least 2” in A2!



Sometimes MORE THAN ONE list of axioms generates the SAME Geometry.

Euclidean is categorical though.

Theorems in the alternate axiom system:

1 Two distinct lines have exactly one point in common.

2 There are exactly 7 points in this geometry.

**Popper 1.1, Question 9**

Why is Fano’s Geometry non-Euclidean?

I. Lines are not made up of an infinite number of points.

II. There are only 7 points total.

III. There are only 7 lines total.

IV. All of the above.

V. None of the above.

A. II and III only

B. I only

C. IV

Enough with finite geometries – there’s an infinite number of them!

In fact, let’s talk about how many there are:

Is there a geometry with 17 points? 1927 points (why did I pick that number?)

N points?

Let’s move on to other geometries with an infinite number of points.

The Klein Disk(Felix Klein, 1849 – 1925, German)

Points will be {(x, y) | x2 + y2 < 1}, the interior of the Unit Circle, and lines will be the set of all lines that intersect the **interior** of this circle. Point, line, and on has the usual Euclidean sense. Incidence axioms are below

So our model is a proper subset of the Euclidean Plane.

Model:

Note that the labeled points (except H) are NOT points in the geometry. A is on the circle not an interior point. It is convenient to use it, though.

H is a point in the circle’s interior and IS a point in the geometry.

We cannot list the number of lines – there are an infinite number of them. A, B, P1, P2, G, C, D, F, and E are not in our space. But the open segments they create are in our space.



Is everybody clear on what is and is not in our space?

**Popper 1.1, Question 10**

In the Klein Disc, lines are made up of an infinite number of points because they are subsets of traditional Euclidean lines.

A. True

B. False

Checking the axioms:

**Undefined terms:** point, line, on

**Axioms:**

IA1 For every pair of distinct points P and Q, there exists exactly one line *l* such that both P and Q lie on that line.

Inheriting…

IA2 For every line *l* there exist at least 2 distinct points P and Q such that both P and Q lie on the line *l*.

Inheriting…

IA3 There exist three points that do not all lie on any one line.

Inheriting…

**Definitions:**

Collinear: Three points, A, B, and C, are said to be collinear if there exists one line l such that all three of the points lie on that line.

Parallel: Lines that share no points are said to be parallel.

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

We will pick line GC and point H. Now look at line GB.



Let’s look at lines GC and GB. They intersect at G…which is NOT a point in the geometry. So GC and GB are parallel. In fact, they are what is called **asymptotically parallel**. They really do share no points in our space.

Now look at P1P2. It, too, is parallel to GC. Furthermore both P1P2 and GB pass through point H. P1P2 is **divergently parallel** to GC.

Not only is the situation vis a vis parallel lines different, we even have flavors of parallel:

**asymptotic and divergent**. So we are truly non-euclidean here, folks.

Theorem 1: If two distinct lines intersect, then the intersection is exactly one point.

Inherited from Euclidean Geometry.

Theorem 2: Each point is on at least two lines.

Each point is on an infinite number of lines.

Theorem 3: There is a triple of lines that do not share a common point.

FE, GC, and AD for example.

Lot’s of Theorems!

Ok now for the other two we’ll be looking at all semester:

Spherical and Hyperbolic

Spherical Geometry is the geometry on the surface of a ball (notably the Earth). Hyperbolic Geometry is the geometry of outer space as far as what we know so far.

They each have an axiomatic system with all the parts to work with. Point, line, plane, on…these are some of the undefined terms for each.

Now Hyperbolic Geometry has 2D models and 3D models just like Euclidean G. Spherical only has 3 D models.

Hyperbolic Geometry

We will look at a (a not the!) 2D model for Hyperbolic G. It’s the unit circle just like in the Klein Disc BUT it has a distance formula that makes the distance inside the disc infinite. Lines are special curves called Orthogononal circles.

Parallel Lines in Hyperbolic Geometry, Poincare disc model. The BOLD circle is not in our space! It’s the interior points only.

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

Pick line AB and Point D.



H is parallel to every other line showing in the disc.

Since H intersects H on the circle, these two have a type of parallelism called “asymptotically parallel”.

H and H are “divergently parallel” to H .

So we have H and a point not on it: Point D and we have 3 lines parallel to H through D right there on the sketch. This illustrates our choice of parallel axiom. And we now have two types of parallelism: asymptotic and divergent, again with this geometry.



I’ll show you many pictures as we move along!

Now let’s look at Spherical Geometry

Spherical G is modeled by the unit sphere and the surface points of the sphere are points in the space. Lines are Great Circles like the equator, but not limited to the equator. Lines through the north and south poles are Great Circles. Grab a ball and 3 rubber bands. Put on an “equator”, then run a band through the north and south pole. Then run the third band through the poles offset from the first one. Note that a plane through a great circle INCLUDES the center of the circle. Other lines of latitude make circles and not lines.

No software available. I am NOT an artist!

Do you see that lines (Great Circles intersect in TWO points? And that more than one line can go through two points? Very Non-Euclidean.

Pick the equator and the North Pole.

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

How many rubber bands go through the North pole and DON’t intersect the equator?

More to come on these two!

**Popper 1.1, Question 11**

There are geometries with no parallel lines with our “pick a line and a point not on that line” setup.

A. True

B. False

**Popper 1.1, Question 12**

Given a line and a point not on that line, some geometries have more than one line through the point sharing no points in the space with the given line.

A. True

B. False

Almost done

**Essay 1.1, Number Two**

What do you think about the fact that multiple geometries exist? Spend some time on the internet researching the history of geometry. Write a 3 paragraph one page, front side only, typed paper (14 point type) discussing how you feel about what you found out. Be sure to put in quotation marks and cite info on anything that you need to.